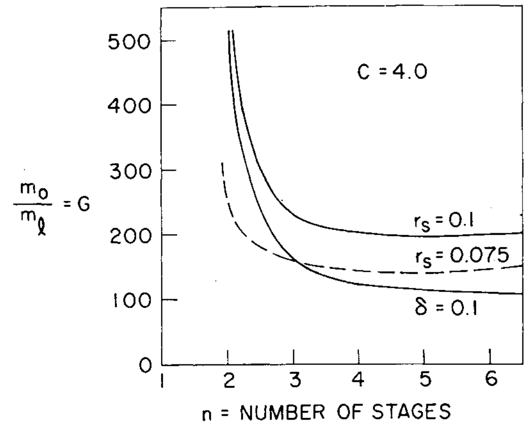

Fig. 2 Variation of mass ratio with different structural factors

number of stages given by $e^{-c/n} - S = 0$. The constant Kc is added as a first linear correction for the horizontal stretching of the knee introduced by an increasing c, with K selected from observation to be 0.35 for typical cases.

It is concluded on the basis of the foregoing that the use of structural factors as functions in optimization analyses introduces the necessity of using either (1) a more definitive structural function when optimizing for minimum mass or (2) a more definitive optimization procedure such as the one suggested above based on the logarithmic nature of the equations. Because of the difficulties of the definitive specification of a structural function, the latter alternative appears preferable.

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On Transient Magnetohydrodynamic Flow in Channels

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In a recent paper Yen and Chang[1] have, using the Laplace transform, studied the magnetohydrodynamic flow in a channel of perfectly conducting walls with a transverse magnetic field and a sudden change of the axial pressure gradient. It is the purpose of this note to investigate the flow in a channel of insulating walls by another method for any change in the time-dependent axial pressure gradient, and also to point out a feature not adequately discussed by Yen and Chang. The velocity field and the induced electric (and magnetic) field are of the oscillatory type with decaying amplitudes, particularly when the magnetic Prandtl number is nearly unity.

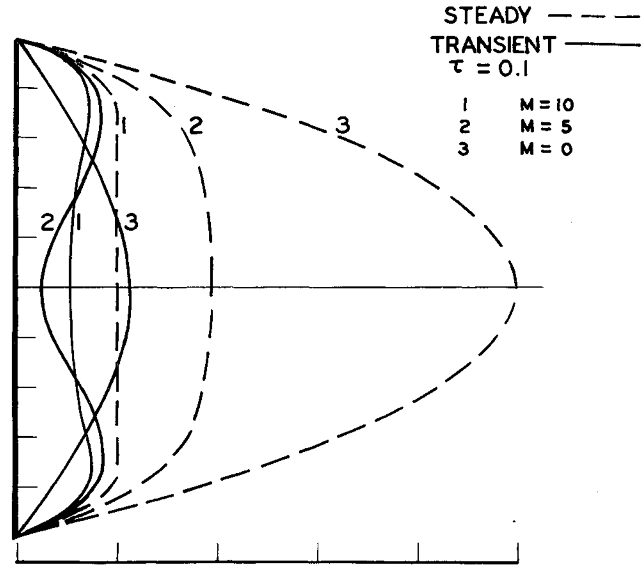

Fig. 1 Velocity profiles with magnetic Prandtl number = 1, $(\rho\nu/P_1h^2)u$ vs $\xi(= y/h)$

Taking x-axis along the channel, the governing equations of one-dimensional unsteady flow are

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} + \frac{P(t)}{\rho} + \frac{\sigma}{\rho} B_0 E_z - \frac{\sigma B_0{}^2}{\rho} u$$

$$\frac{\partial E_z}{\partial t} + B_0 \frac{\partial u}{\partial t} = \lambda \frac{\partial^2 E_z}{\partial y^2} \tag{1}$$

$$\frac{\partial B_x}{\partial t} - \lambda \frac{\partial^2 B_x}{\partial y^2} = B_0 \frac{\partial u}{\partial y}$$

where P is the time-dependent axial pressure gradient. The initial and boundary conditions are

$$t = 0: \quad u = u_s(y),\ P = P_s = \text{const},\ E_z = E_s = \text{const}$$
$$t > 0;\ y = \pm h: \quad u = 0,\ P = P_1(t) + P_s,\ E_z = E_1(t) + E_s \tag{2}$$

where the subscript s denotes the steady state. E_s and E_1 are unknown a priori, they must be found from the integration of current density,

$$t \geqslant 0: \quad \int_{-h}^{h} j_z\, dy = \int_{-h}^{h} (E_z + uB_0)\, dy = 0 \tag{3}$$

However, owing to the property of symmetry the instantaneous electric field at both walls $y = \pm h$ must be the same.

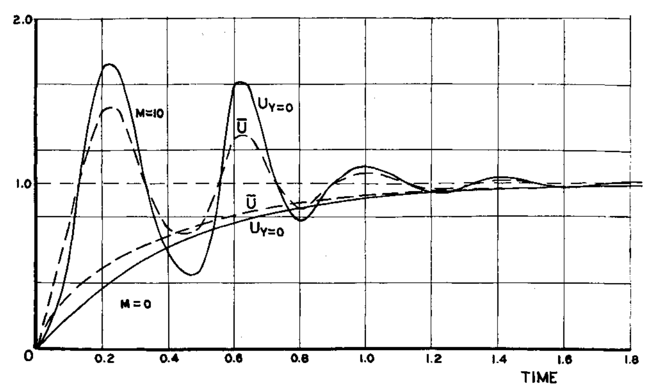

Fig. 2 Time history of $U = u/u_s$ (magnetic Prandtl number = 1)

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Since the first two equations of Eqs. (1) are independent of B_x, which can be found after u and E_z are determined, we will only study the solutions of u and E_z. Similar to the method used in a previous paper,[2] we use Duhamel's theorem for a system of coupled equations and find that the solutions for any given pressure variation are

$$u = \frac{\cosh M - \cosh M\xi}{\sinh M}\,\frac{M}{\sigma B_0{}^2}\left[P_s + \frac{\partial}{\partial t}\int_0^t P_1(t')dt'\right] + \frac{h^2}{\rho\nu}\sum_0 \cos\frac{(2n+1)\pi\xi}{2}\,\frac{\partial}{\partial \tau}\int_0^\tau P_1(\tau') \times \exp\left[-\frac{(2n+1)^2\pi^2}{8}\left(1+\frac{\lambda}{\nu}\right)(\tau-\tau')\right]\times \{A_{2n+1}\times \sin\,[R_{2n+1}(\tau-\tau')] + B_{2n+1}\cos[R_{2n+1}(\tau-\tau')]\}d\tau' \quad (4)$$

$$E_z = \frac{1 - M\coth M}{\sigma B_0}\left[P_s + \frac{\partial}{\partial t}\int_0^t P_1(t')dt'\right] + \frac{1}{\sigma B_0}\times \sum_0 \cos\frac{(2n+1)\pi\xi}{2}\,\frac{\partial}{\partial \tau}\int_0^\tau P_1(\tau')\exp\left[-\frac{(2n+1)^2\pi^2}{8}\times\left(1+\frac{\lambda}{\nu}\right)(\tau-\tau')\right]\left\{\left\langle\left[\frac{(2n+1)^2\pi^2}{8}\left(\frac{\lambda}{\nu}-1\right) - M^2\right]A_{2n+1} + R_{2n+1}B_{2n+1}\right\rangle \sin[R_{2n+1}(\tau-\tau')] + \left\langle\left[\frac{(2n+1)^2\pi^2}{8}\left(\frac{\lambda}{\nu}-1\right) - M^2\right]B_{2n+1} - R_{2n+1}A_{2n+1}\right\rangle \cos[R_{2n+1}(\tau-\tau')]\right\}d\tau' \quad (5)$$

where

$$\xi = y/h, \quad \tau = \nu t/h^2, \quad M = \sigma B_0{}^2h^2/\rho\nu$$

$$R_n = \frac{n\pi}{2}\left[\frac{\lambda}{\nu}M^2 - \frac{n^2\pi^2}{16}\left(1-\frac{\lambda}{\nu}\right)^2\right]^{1/2}$$

$$A_{2n+1} = \frac{2}{R_{2n+1}}\,\frac{(-1)^n}{(2n+1)\pi}\left[\frac{(1+\lambda/\nu)(2n+1)^2\pi^2}{4M^2+(2n+1)^2\pi^2}\times M\coth M - 2\right] \quad (6)$$

$$B_{2n+1} = \frac{(-1)^{n+1}}{(2n+1)\pi[4M^2+(2n+1)^2\pi^2]}\,16M\coth M$$

Special Cases

1. Sudden Start

We consider the problem that the fluid is initially at rest and is set in motion by a sudden application of a constant axial pressure gradient. This readily implies that $P_s = 0$ and $P_1 = \text{const}$. Performing the integrations, we find

$$u = \frac{P_1M}{\sigma B_0{}^2}\left[\frac{\cosh M - \cosh M\xi}{\sinh M}\right] + \frac{P_1h^2}{\rho\nu}\times \sum_0 \exp\left[-\frac{(2n+1)^2\pi^2}{8}\left(1+\frac{\lambda}{\nu}\right)\tau\right][A_{2n+1}\sin R_{2n+1}\tau + B_{2n+1}\cos R_{2n+1}\tau]\cos\frac{(2n+1)\pi\xi}{2} \quad (7)$$

$$E_z = \frac{P_1}{\sigma B_0}(1 - M\coth M) + \frac{P_1}{\sigma B_0}\sum_0 \exp\left[-\frac{(2n+1)^2\pi^2}{8}\times\left(1+\frac{\lambda}{\nu}\right)\tau\right]\left\{\left[\left(\frac{(2n+1)^2\pi^2}{8}\left(\frac{\lambda}{\nu}-1\right) - M^2\right)A_{2n+1} + R_{2n+1}B_{2n+1}\right]\sin R_{2n+1}\tau + \left[\left(\frac{(2n+1)^2\pi^2}{8}\left(\frac{\lambda}{\nu}-1\right) - M^2\right)B_{2n+1} - R_{2n+1}A_{2n+1}\right]\cos R_{2n+1}\tau\right\}\cos\frac{(2n+1)\pi\xi}{2} \quad (8)$$

As $t \to \infty$, we may recover the steady Hartmann problem.[3] Also we may recover the solution in a nonmagnetic field by letting $M \to 0$.

2. Sudden Removal of the Pressure Gradient

A steady state is suddenly changed by the removal of the pressure gradient. This states that $P_1 = -P_s$. The solutions become

$$u = -\frac{P_sh^2}{\rho\nu}\sum_0 \exp\left[-\frac{(2n+1)^2\pi^2}{8}\left(1+\frac{\lambda}{\nu}\right)\tau\right]\times [A_{2n+1}\sin R_{2n+1}\tau + B_{2n+1}\cos R_{2n+1}\tau]\cos\frac{(2n+1)\pi\xi}{2} \quad (9)$$

$$E_z = -\frac{P_s}{\sigma B_0}\sum_0 \exp\left[-\frac{(2n+1)^2\pi^2}{8}\left(1+\frac{\lambda}{\nu}\right)\tau\right]\times \left\{\left[\left(\frac{(2n+1)^2\pi^2}{8}\left(\frac{\lambda}{\nu}-1\right) - M^2\right)A_{2n+1} + R_{2n+1}B_{2n+1}\right]\sin R_{2n+1}\tau + \left[\left(\frac{(2n+1)^2\pi^2}{8}\left(\frac{\lambda}{\nu}-1\right) - M^2\right)B_{2n+1} - R_{2n+1}A_{2n+1}\right]\cos R_{2n+1}\tau\right\}\cos\frac{(2n+1)\pi}{2}\xi \quad (10)$$

To illustrate the oscillatory nature of the problem, we present some numerical results of the case of sudden start with magnetic Prandtl number $\lambda/\nu = 1$. Figure 1 shows the velocity profiles of different Hartmann numbers in dimensionless velocity $(\rho\nu/P_1h^2)u$ at the dimensionless time $\tau = \nu t/h^2 = 0.1$. Figure 2 gives the time history of the velocity magnitudes at the center of the channel and that of the average velocity.

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The Use of Macauley's Brackets in the Analysis of Laterally Loaded Struts and Tie-Bars

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Nomenclature

a = x-coordinate locating lateral load
C_1, C_2 = integration constants
EI = flexural rigidity
k = $(P/EI)^{1/2}$
L = span
M = bending moment

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